Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 6, pp. 934–945, 2008

PLANE OVSYANNIKOV VORTEX: MOTION PROPERTIES AND EXACT SOLUTIONS

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UDC 533

The physical properties of ideal plasma flow described by the Ovsyannikov plane vortex are studied. The particle trajectories and magnetic lines are shown to be plane curves, and an algorithm for describing the motion in three-dimensional space is proposed. Some exact solutions of the submodel are obtained and studied.

Key words: *ideal magnetohydrodynamics, exact solutions, trajectories, magnetic lines, uniform deformation.*

A partially invariant submodel of the equations of ideal magnetic hydrodynamics that defines three-dimensional motion of a continuous medium with plane waves similar to the Ovsyannikov spherical vortex [2, 3] was constructed in [1]. The equations of the submodel were derived and analyzed, and a geometrical algorithm for searching the noninvariant function included in the solution was proposed. The present work is a continuation of [1].

1. Equations of the Submodel. We study the model of ideal magnetohydrodynamics [4]

$$
\begin{gathered}
D\rho + \rho \operatorname{div} \boldsymbol{u} = 0, \qquad D\boldsymbol{u} + \rho^{-1}\nabla p + \rho^{-1}\boldsymbol{H} \times \operatorname{rot} \boldsymbol{H} = 0, \\
Dp + A(p,\rho) \operatorname{div} \boldsymbol{u} = 0, \qquad D\boldsymbol{H} + \boldsymbol{H} \operatorname{div} \boldsymbol{u} - (\boldsymbol{H} \cdot \nabla)\boldsymbol{u} = 0, \\
\operatorname{div} \boldsymbol{H} = 0, \qquad D = \partial_t + \boldsymbol{u} \cdot \nabla,
\end{gathered}
\tag{1.1}
$$

where $\boldsymbol{u} = (u, v, w)$ is the velocity vector, $\boldsymbol{H} = (H, K, L)$ is the magnetic field intensity vector, and p and ρ are the pressure and density, respectively. The equation of state $p = F(S, \rho)$ with entropy S holds. The function $A(p, \rho)$ is given by the equation of state $A = \rho\,(\partial F/\partial\rho)$. All functions depend on time t and Cartesian coordinates $\boldsymbol{x} = (x, y, z)$.

The partially invariant solution of [5] system (1.1) is written as [1]

$$
\begin{aligned}
&u = U(t,x), && H = H(t,x), \\
&v = V(t,x)\cos\omega(t,x,y,z), && K = N(t,x)\cos\omega(t,x,y,z), \\
&w = V(t,x)\sin\omega(t,x,y,z), && L = N(t,x)\sin\omega(t,x,y,z), \\
&p = p(t,x), \qquad \rho = \rho(t,x), && S = S(t,x).
\end{aligned}
\tag{1.2}
$$

The functions depending only on t and x will further be called invariant; the unique noninvariant function is the function ω. In [1], an auxiliary invariant function h was introduced using the equality

$$
\tilde{D}\rho + \rho(U_x + hV) = 0, \qquad \tilde{D} = \partial_t + U\partial_x.
$$

Two cases are distinguished: $h = 0$ and $h \neq 0$. In the case $h \neq 0$, for convenience, we introduce the function $\tau = 1/h$; then, the subsystem for the invariant functions becomes

$$
\tilde{D}\rho + \rho(U_x + \tau^{-1}V) = 0, \qquad \tilde{D}U + \rho^{-1}p_x + \rho^{-1}NN_x = 0,
$$

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$$\tilde{D}V - \rho^{-1}H_0\tau^{-1}N_x = 0, \qquad \tilde{D}p + A(p,\rho)(U_x + \tau^{-1}V) = 0, \tag{1.3}$$

$$\tilde{D}N + NU_x - H_0\tau^{-1}V_x + \tau^{-1}NV = 0, \qquad \tilde{D}\tau = V, \qquad H_0\tau_x = \tau N.$$

The noninvariant function ω is determined from the implicit finite relation

$$F(y - \tau\cos\omega, z - \tau\sin\omega) = 0 \tag{1.4}$$

with an arbitrary smooth function F.

In the case $h = 0$, the subsystem for the invariant functions becomes

$$\tilde{D}\rho + \rho U_x = 0, \qquad \tilde{D}U + \rho^{-1}p_x + \rho^{-1}NN_x = 0,$$

$$\tilde{D}V - \rho^{-1}H_0N_x = 0, \qquad \tilde{D}p + A(p,\rho)U_x = 0, \tag{1.5}$$

$$\tilde{D}N + NU_x - H_0V_x = 0, \qquad \tilde{D}\varphi = V, \qquad H_0\varphi_x = N.$$

The implicit relation for the function ω is written as

$$y\cos\omega + z\sin\omega = f(\omega) + \varphi(t,x), \tag{1.6}$$

where f is an arbitrary smooth function. In [1], geometrical algorithms for solving the implicit equations (1.4) and (1.6) were proposed, the domains of the solution were found, and possible cases of singularities were considered. Below, the general picture of motion in three-dimensional space is studied.

2. Particle Trajectories and Magnetic Lines. Differentiation of Eqs. (1.4) and (1.6) yields the equality

$$D\omega = 0,$$

where D is the operator of total differentiation along the trajectory. The angle ω is conserved along the trajectory, and, hence, this trajectory lies entirely in a certain plane which is parallel to the Ox axis and is turned by angle ω about it. Differentiation along the magnetic lines shows that ω is also conserved along these curves. Thus, for each particle, its trajectory and magnetic line are plane curves belonging to the same plane determined by the angle ω.

Another important property of the solution follows from representation (1.2). To determine the trajectory of a certain particle, we pose the Cauchy problem. The plane motion of a particle is completely determined by the velocity components U and V dependent only on the invariant variables t and x; therefore, for an arbitrary two particles belonging to a certain plane $x = x_0$ at the initial time $t = t_0$, the Cauchy problems for the trajectories coincide. Although different particles move in different planes, their trajectories, as plane curves, are identical. The same is true for the magnetic lines through two different points of the same plane $x = x_0$. Thus, it is possible to construct the trajectory and magnetic-line patterns for the particles lying in the plane $x = x_0$. Attaching these patterns to each point of the plane $x = x_0$ in the domain of definition of the function ω according to the direction field determined by the function ω, we obtain a three-dimensional pattern of the trajectories and magnetic lines over the entire space (Fig. 1).

To construct the pattern, we consider the plane of motion of a certain particle located at a point $M = (x_0, y_0, z_0)$ at the initial time $t = t_0$. The plane considered is parallel to the Ox axis and is turned about it by an angle ω relative to the Oy axis. In this plane, the Cartesian coordinate system is defined as follows. The coordinate origin O' is placed at the point of the orthogonal projection of M onto the plane Oyz. One of the coordinate axes is parallel to the Ox axis and is also denoted by x. The $O'l$ axis is orthogonal to the $O'x$ axis, so that the coordinate system $O'xl$ had the right orientation (see Fig. 1). In this coordinate system, the particle trajectory is determined by solving the following Cauchy problem:

$$\frac{dx}{dt} = U(t,x), \qquad x(t_0) = x_0. \tag{2.1}$$

Solution of problem (2.1) gives the dependence $x = x(t, x_0)$, which is integrated to yield the dependence $l = l(t)$:

$$l(t) = \int_{t_0}^{t} V(t, x(t,x_0))\,dt. \tag{2.2}$$

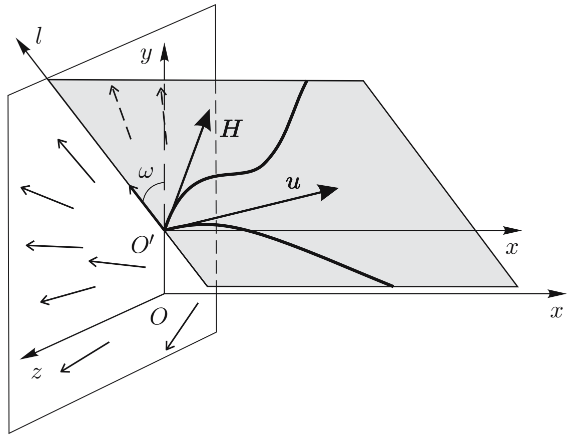

Fig. 1. Spatial picture of motion.

Finally, in the initial coordinate system $Oxyz$, the equations of the particle trajectory are retrieved in the form

$$x = x(t, x_0), \qquad y = y_0 + l(t)\cos\omega_0, \qquad z = z_0 + l(t)\sin\omega_0, \tag{2.3}$$

where $\omega_0 = \omega(t_0, \boldsymbol{x}_0)$ is the value of the angle ω calculated according to the implicit relation (1.4) at the point M at the initial time.

The magnetic line at the time $t = t_0$ is defined as the integral curve of the magnetic field. The expressions defining the magnetic line through the point $M = (x_0, y_0, z_0)$ at the time $t = t_0$ are written as

$$y = y_0 + \cos\omega_0 \int\limits_{x_0}^{x} \frac{N(t_0, s)}{H(t_0, s)}\, ds, \qquad z = z_0 + \sin\omega_0 \int\limits_{x_0}^{x} \frac{N(t_0, s)}{H(t_0, s)}\, ds. \tag{2.4}$$

Equations (2.4) are derived similarly to relations (2.3). Thus, the following theorem holds.

Theorem 1. *The plasma motion given by solution* (1.2) (*see Fig. 1*) *has the following properties:*

1. The trajectories and magnetic lines are plane curves and lie in the planes which are parallel to the Ox axis and are turned by angle ω about it relative to the positive Oy direction.

2. All particles that belong to a plane $x = x_0$ at a time $t = t_0$ circumscribe identical trajectories in the planes of their motion. The magnetic lines through points of the plane $x = x_0$ are also plane curves which belong to the same planes as the trajectories of the corresponding particles of the plane $x = x_0$.

We note an interesting property of the solutions described by the given submodel. By varying the direction fields in the plane $x = x_0$, it is possible to construct an infinite set of pictures of motion using the same pattern of trajectories and magnetic lines. The geometrical algorithms for constructing the direction field obtained in [1] allow the picture of motion to be modified according to required characteristics of the motion described. The same property holds for the spherical Ovsyannikov vortex [3].

3. Domain of the Solution in Three-Dimensional Space. The above constructions lead to an algorithm for searching the total domain of definition of the solution in three-dimensional space. In a fixed plane $x = x_0$, the function ω has a domain of definition (finite in many cases) which is bounded by τ-equidistants to γ in the case $h \neq 0$ and by the curve described in [1] in the case $h = 0$. In both cases, the direction field given by the function ω in the plane $x = x_0$ is orthogonal to the boundary of the domain of the function ω. To find the boundaries of the domain of the solution in three-dimensional space, it is necessary to attach the magnetic line pattern to each point of the boundary of the domain of the function ω in the plane $x = x_0$. As a result, we obtain a channel whose walls are "woven" from magnetic lines. Because of the freezing-in property of the magnetic lines, the channel walls can be treated as impermeable infinitely conducting pistons. In the case of stationary solutions, the walls are fixed; in the nonstationary case, the walls are extended or shrunk according to the behavior of the function τ ($h \neq 0$) and the function φ ($h = 0$). In the case of a finite domain of the function ω (this domain can always be bounded to a finite one), each section of the total domain of the solution by a plane orthogonal to the Ox axis is finite, and, hence, the kinetic and magnetic energies are also finite in this domain.

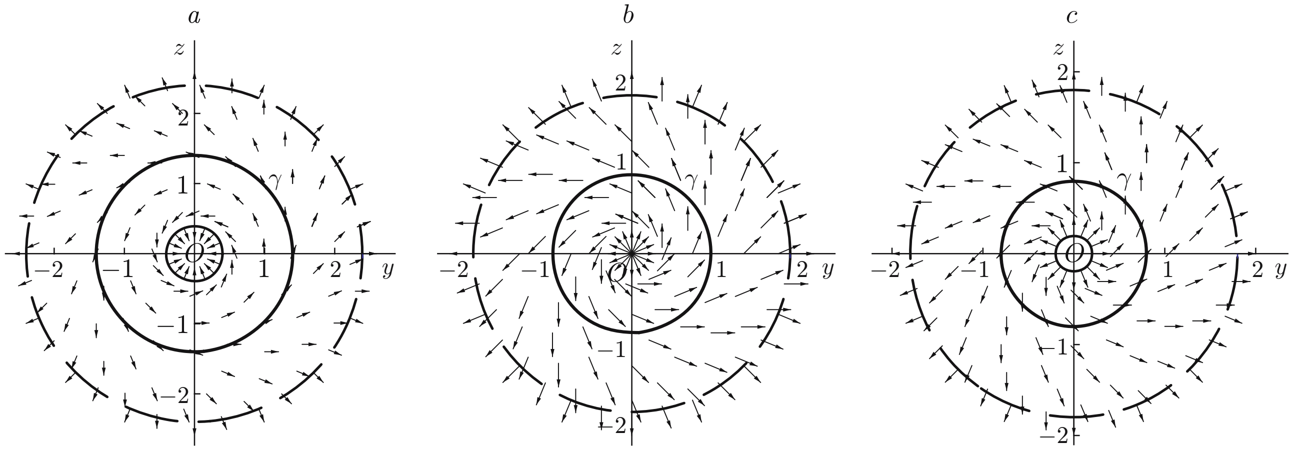

Fig. 2. Direction field given by the function ω according to Eq. (1.4): $R > \tau$ (a), $R = \tau$ (b), and $R < \tau$ (c).

4. Example of Stationary Solution. As an example we consider the simplest stationary solution of system (1.3) given by the formulas

$$U = H_0^2 \sinh x, \qquad V = H_0^2 \tanh x, \qquad \tau = \cosh x,$$
$$H = H_0 \sinh x, \qquad N = H_0 \tanh x, \qquad \rho = H_0^{-2}, \qquad S = S_0. \tag{4.1}$$

In this solution (as well as in all stationary solutions of the type described), the velocity and magnetic field intensity vector are collinear at each particle. Solution (4.1) is a special case of Chandrasekhar's solution [6]. The streamlines and magnetic lines coincide, and for $x_0 = 0$ and

$$l(x) = \cosh x - 1, \tag{4.2}$$

they are given by Eqs. (2.3). The streamlines are parts of a catenary. We note that solution (4.1) can be continuously conjugate with the uniform flow along the Ox axis. Indeed, in the section $x = 0$, all functions in (4.1) and their derivatives take values corresponding to the uniform flow. We construct a solution which switches the uniform flow to the plane Ovsyannikov vortex (4.1) at the section $x = 0$.

The pattern of the streamlines and magnetic lines is a straight line which is parallel to the Ox axis at $x < 0$ and smoothly becomes curve (4.2) at $x \geqslant 0$. In Eq. (1.4), we choose the function $F(y, z) = y^2 + z^2 - R^2$. In this case, the curve γ: $F(y, z) = 0$ is a circle. Figure 2 shows the direction fields obtained for various relation between τ and R according to the algorithm of [1]. For $R > \tau$, the direction field is given in the circular zone of definition between two circles of radii $R \pm \tau$ in the plane Oyz. On the inner circle $|\boldsymbol{x}| = R - \tau$, the field is directed toward the point O. In the case $R = \tau$, the inner circle shrinks to the point O. In this case, the direction field at this point is not unique. Finally, for $R < \tau$, the inner circle is inverted and becomes a circle of radius $\tau - R$ with the direction field oriented into the zone of definition. These direction fields generate various pictures of motion in three-dimensional space.

The streamline pattern described above is attached to each point of the plane Oyz in the domain according to the direction field shown in Fig. 2. By virtue of the obvious central symmetry of the direction field, the obtained picture of motion is axisymmetric. Figure 3 shows the axial section of the region of three-dimensional space occupied by the motion. Depending on the relation between $\tau(0)$ and R, three different pictures of motion are possible. Each particle moves along a plane curve, but the orientation of the curves in the space depends on the chosen particle. At the section $x = 0$, the uniform flow in a cylindrical channel with central body at $x < 0$ becomes the curvilinear channel flow at $x > 0$ described by solution (4.1). The cases presented in Fig. 3a–c correspond to the direction fields shown in Fig. 2a–c. Three-dimensional visualization of motion at $R > \tau(0)$ is presented in Fig. 4, which shows fragments of the channel walls and the particle streamlines. It is evident that each streamline has the same plane curve shape. The orientation of the streamlines is determined by the direction field in Fig. 2a, and the axial section of the channel is presented in Fig. 3a.

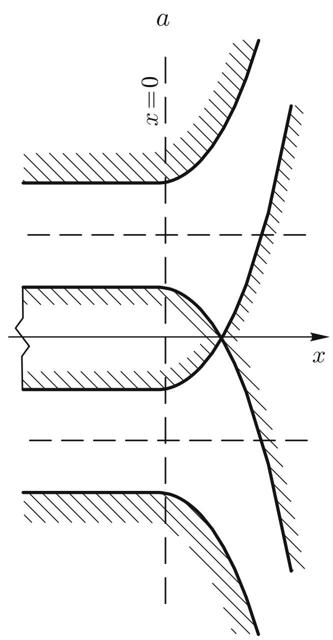

Fig. 3. Axial sections of the channels occupied by the gas flow: (a) $R > \tau(0)$; (b) $R = \tau(0)$; (c) $R < \tau(0)$.

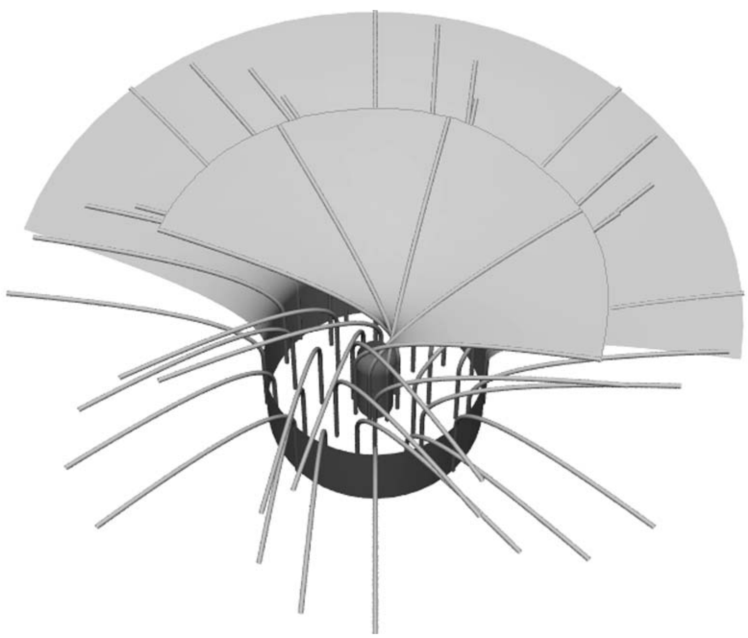

Fig. 4. Spatial flow visualization.

5. Solutions with Uniform Deformation. We consider the solutions of the invariant subsystems (1.3) and (1.5) in which the velocity component U depends linearly on the spatial coordinate x. For the equations of magnetohydrodynamics (1.1) assuming a linear dependence of all velocity components on all spatial coordinates, such motions were studied in [7].

For convenience, instead of (t, x) we use the Lagrangian coordinates (t, ξ) where ξ satisfies the equations

$$\xi_t + U\xi_x = 0, \qquad \xi(0, x) = x. \tag{5.1}$$

We introduce the function $M = \partial x/\partial\xi$. In transformation to the Lagrangian coordinates, the derivatives of the arbitrary function $f(t, x)$ are converted as follows:

$$f_t + Uf_x \to f_t, \qquad f_x \to M^{-1}f_\xi.$$

In addition, by the definition of the Lagrangian coordinates, we have

$$U = x_t, \qquad U_x = M^{-1}M_t.$$

We first study system (1.5).

Making the change of variables in system (1.5), we obtain

$$\begin{gathered} \rho_t + M^{-1}\rho M_t = 0, \qquad x_{tt} + \rho^{-1}M^{-1}(p_\xi + NN_\xi) = 0, \\ V_t - \rho^{-1}M^{-1}H_0N_\xi = 0, \qquad p_t + M^{-1}A(p, \rho)M_t = 0, \\ N_t + M^{-1}NM_t - H_0M^{-1}V_\xi = 0, \qquad \varphi_t = V, \qquad H_0\varphi_\xi = MN. \end{gathered} \tag{5.2}$$

System (5.2) has two obvious integrals

$$\rho M = f(\xi), \qquad S = S(\xi) \tag{5.3}$$

(S is the entropy). In view of (5.3), from system (5.2) we obtain

$$(MN)_t = H_0V_\xi, \qquad f(\xi)V_t = H_0N_\xi, \qquad x_{tt} + f(\xi)^{-1}(p_\xi + NN_\xi) = 0. \tag{5.4}$$

From the last two equations of system (5.2), V and N can be expressed as functions of φ:

$$V = \varphi_t, \qquad N = H_0M^{-1}\varphi_\xi. \tag{5.5}$$

Substitution of expressions (5.5) into the first equation (5.4) yields the identity, i.e., the first equation (5.4) is the condition of compatibility of Eqs. (5.5) with respect to the function φ. The second relation (5.4) leads to the linear equation for φ:

$$f(\xi)\varphi_{tt} = H_0^2(M^{-1}\varphi_\xi)_\xi. \tag{5.6}$$

Let us make the following assumptions: 1) the gas is polytropic; i.e., $p = S(\xi)\rho^\gamma$; 2) the dependence $x(\xi)$ is linear, i.e.,

$$x = M(t)\xi, \qquad M(0) = 1, \qquad U = \dot{M}(t)\xi. \tag{5.7}$$

Here and below, dot above a symbol denotes the derivative with respect to time t and prime denotes the derivative with respect to ξ. For continuous motion of a continuous medium, the function M cannot vanish. For definiteness, we assume that $M > 0$. Under the above assumptions, the equations of the system reduce to the following key relation:

$$\ddot{M}\xi + f(\xi)^{-1}\Big(\frac{(S(\xi)f(\xi)^\gamma)'}{M(t)^\gamma} + H_0^2\frac{\varphi_\xi\varphi_{\xi\xi}}{M(t)^2}\Big) = 0. \tag{5.8}$$

To separate the variables in Eq. (5.8), we assume that the function φ has the form

$$\varphi(t, \xi) = \alpha(t)\beta(\xi).$$

From Eq. (5.6), we obtain

$$f(\xi)M(t)\ddot{\alpha}(t)\beta(\xi) = H_0^2\alpha(t)\beta''(\xi). \tag{5.9}$$

Separating the variables in Eq. (5.9), we have

$$\frac{M(t)\ddot{\alpha}(t)}{\alpha(t)} = \frac{H_0^2\beta''(\xi)}{f(\xi)\beta(\xi)} = C_1, \tag{5.10}$$

where C_1 is an arbitrary constant. Next, from Eq. (5.8), we have

$$\ddot{M}\xi + f(\xi)^{-1}\Big(\frac{\big(S(\xi)f(\xi)^\gamma\big)'}{M(t)^\gamma} + H_0^2\frac{\alpha(t)^2\beta'(\xi)\beta''(\xi)}{M(t)^2}\Big) = 0. \tag{5.11}$$

Equation (5.11) is written as the scalar product $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, where

$$\boldsymbol{a} = (\ddot{M}, M^{-\gamma}, H_0^2\alpha^2 M^{-2}), \qquad \boldsymbol{b} = (\xi, (Sf^\gamma)'f^{-1}, \beta'\beta'' f^{-1}).$$

In this equation, the variables are separated according to the Ovsyannikov lemma [8]. Here the unique nontrivial case ($M \neq \text{const}$) is the following one:

$$\frac{(S(\xi)f(\xi)^\gamma)'}{f(\xi)} = C_2\xi, \qquad \frac{\beta'(\xi)\beta''(\xi)}{f(\xi)} = C_3\xi; \tag{5.12}$$

$$\ddot{M} + \frac{C_2}{M^\gamma} + H_0^2\,\frac{C_3\alpha^2}{M^2} = 0 \tag{5.13}$$

(C_2 and C_3 are arbitrary constants). From the second equation (5.12) and the second equation (5.10), we obtain

$$\beta'\beta'' = C_3\xi f, \qquad H_0^2\beta'' = C_1 f\beta. \tag{5.14}$$

Dividing the first equation (5.14) by the second, we find that

$$\frac{\beta'}{H_0^2} = \frac{C_3\xi}{C_1\beta} \qquad \Longrightarrow \qquad \beta^2 = \frac{C_3}{C_1}H_0^2\,\xi^2 + C_4 \tag{5.15}$$

(C_4 is an arbitrary constant). From the remaining relation (5.14), we have

$$f(\xi) = \frac{H_0^2\beta''}{C_1\beta} = \frac{C_3C_4H_0^4}{(C_3H_0^2\,\xi^2 + C_1C_4)^2}. \tag{5.16}$$

By virtue of (5.3) and the above assumption of nonnegative function M, from expression (5.16) we obtain the constraint $C_3C_4 > 0$. The first equation (5.12) serves to determine the unknown function $S(\xi)$:

$$S(\xi) = f(\xi)^{-\gamma}\Big(S_0 + C_2\int \xi f(\xi)\,d\xi\Big).$$

Substitution of the expressions for $S(\xi)$ and $\rho = fM^{-1}$ into the equation of state $p = S\rho^\gamma$ yields the pressure

$$p = \frac{1}{M^\gamma}\Big(p_0 - \frac{C_2C_4H_0^2}{2(C_3H_0^2\xi^2 + C_1C_4)}\Big).$$

From (5.10) and (5.13), we determine the functions M and α dependent on t:

$$\ddot{M} = -\frac{C_2}{M^\gamma} - H_0^2\frac{C_3\alpha^2}{M^2}, \qquad \ddot{\alpha} = \frac{C_1\alpha}{M}. \tag{5.17}$$

System (5.17) is a closed system of ordinary differential equations for the functions M and α, which will be studied below.

Let us determine the equations for the particle trajectories and magnetic lines.

The definition of the Lagrange coordinates ξ leads to the dependence $x(t)$ along the particle trajectory:

$$x = x_0M(t). \tag{5.18}$$

In view of (2.2) and (5.5), the expression for the coordinate $l(t)$ becomes

$$l(t) = \int\limits_0^t V(s, x(s,x_0))\,ds = \int\limits_0^t \dot{\alpha}(s)\beta(x_0)\,ds = (\alpha(t) - \alpha(0))\beta(x_0). \tag{5.19}$$

Finally, the particle trajectory in three-dimensional space is found from formulas (2.3). To construct the magnetic lines at the time $t = t_0$ for the given solution, it is necessary to calculate the integral

$$\int\limits_{x_0}^{x} \frac{N(t_0, s)}{H(t_0, s)}\, ds = \int\limits_{x_0}^{x} \frac{H_0\alpha(t)\beta'(sM^{-1})}{H_0 M(t)}\, ds = \alpha(t)(\beta(xM^{-1}) - \beta(x_0)). \tag{5.20}$$

The magnetic lines are given by formulas (2.4). Thus, the functions $M(t)$ and $\alpha(t)$ define the particle trajectory in parametric form, and the function $\beta(\xi)$ gives the magnetic line shape.

From Eq. (5.15) for the function β, it follows that there are two significantly different cases: $C_1C_3 > 0$ and $C_1C_3 < 0$. In the case $C_1C_3 > 0$, Eq. (5.15) defines a family of hyperbolas in the plane (ξ, β). This implies that the domain of existence of the solution has no limit along the Ox axis. In the case $C_1C_3 < 0$, Eq. (5.15) defines a family of ellipses and, hence, the solution is determined only for $|\xi| < \sqrt{C_1C_4/(C_3H_0^2)}$. The second case is physically meaningless since on the boundaries of the domain of the solution, the density and pressure tends to infinity. Next we assume that $C_1C_3 > 0$. In addition, the function β is defined and different from zero for all ξ if the inequality $C_4 > 0$ holds. Thus, the following conditions should be satisfied:

$$C_1 > 0, \qquad C_3 > 0, \qquad C_4 > 0, \qquad p_0 > C_2H_0^2/(2C_1). \tag{5.21}$$

The sign of the constant C_2 determines the dependence of the pressure p on the Lagrangian coordinate ξ. At infinity $\xi \to \infty$, the pressure depends only on time and is equal to $p_0M^{-\gamma}$. For $C_2 < 0$, the pressure at the coordinate origin is lower than that at infinity, i.e., the solution describes the gas acceleration under the action of the internal pressure. For $C_2 > 0$, the pressure at the coordinate origin is lower than that at infinity, i.e., the motion of gas occurs under the action of the elevated external pressure.

We return to system (5.17). Since stretching allows the function α to be determined to within an arbitrary constant factor, it is possible to choose $C_3 = C_1(2H_0^2)^{-1}$. Then, $\beta^2 = \xi^2/2 + C_4$. The magnetic lines are hyperbolas. The dynamic system (5.17) becomes

$$\ddot{M} = -\frac{C_2}{M^\gamma} - \frac{C_1\alpha^2}{2M^2}, \qquad \ddot{\alpha} = \frac{C_1\alpha}{M}. \tag{5.22}$$

System (5.22) can be written in the form of the Lagrange equation with the Lagrangian

$$L = \frac{\dot{M}^2 + \dot{\alpha}^2}{2} + \frac{C_2}{(\gamma-1)M^{\gamma-1}} + \frac{C_1\alpha^2}{2M}.$$

For this system, the energy integral holds:

$$\frac{\dot{M}^2 + \dot{\alpha}^2}{2} - \frac{C_2}{(\gamma-1)M^{\gamma-1}} - \frac{C_1\alpha^2}{2M} = b.$$

Denoting the derivatives as

$$\dot{M} = r\cos\theta, \qquad \dot{\alpha} = r\sin\theta,$$

from the energy integral, we obtain the expression

$$r = \sqrt{2b + \frac{C_1\alpha^2}{M} + \frac{2C_2}{(\gamma-1)M^{\gamma-1}}}.$$

Then, system (5.22) is written as

$$r\dot{\theta} = \frac{C_1\alpha}{M}\cos\theta + \Big(\frac{C_2}{M^\gamma} + \frac{C_1\alpha^2}{2M^2}\Big)\sin\theta, \qquad \dot{M} = r\cos\theta, \qquad \dot{\alpha} = r\sin\theta. \tag{5.23}$$

Further analysis of system (5.23) can be performed numerically. The results of the study are summarized in the following theorem.

Theorem 2. *The solution of Eqs.* (1.5) *with a linear dependence* $U(x)$ *is given by the formulas*

$$U = \frac{\dot{M}(t)}{M(t)}\, x, \qquad V = \dot{\alpha}(t)\beta(\xi), \qquad N = \frac{H_0\xi\alpha(t)}{2M(t)\beta(\xi)}, \qquad \beta = \sqrt{\frac{\xi^2}{2} + C_4}, \qquad \xi = \frac{x}{M(t)},$$

$$p = \frac{1}{M(t)^\gamma}\Big(p_0 - \frac{C_2C_4H_0^2}{C_1(\xi^2+2C_4)}\Big), \qquad \rho = \frac{2C_4H_0^2}{C_1M(t)(\xi^2+2C_4)^2}, \qquad \varphi = \alpha(t)\beta(\xi)$$

with arbitrary constants C_i and p_0 satisfying inequalities (5.21)*. The functions M and α are determined from the ordinary differential equations* (5.23)*. For this solution, the magnetic lines are hyperbolas and are given by formulas* (2.4) *and* (5.20)*. The particle trajectories are given by Eqs.* (2.3)*, and the dependence $l(t)$ is defined in* (5.19)*.*

For the main case $h \neq 0$, the submodel with a linear dependence $U(x)$ is studied similarly. Using the function $\tau = 1/h$ instead of h, we write Eqs. (1.3) in the Lagrangian coordinates (5.1):

$$\tau(M\rho)_t + M\rho V = 0; \tag{5.24}$$

$$x_{tt} + \rho^{-1}M^{-1}(p_\xi + NN_\xi) = 0; \tag{5.25}$$

$$\rho M\tau V_t - H_0N_\xi = 0; \tag{5.26}$$

$$M\tau p_t + \gamma p(\tau M_t + MV) = 0; \tag{5.27}$$

$$\tau(MN)_t - H_0V_\xi + MNV = 0; \tag{5.28}$$

$$\tau_t = V, \qquad H_0\tau_\xi = MN\tau. \tag{5.29}$$

Equations (5.24) and (5.27), together with the equation of state $p = S\rho^\gamma$, lead to the first integrals of the system

$$\tau M\rho = f(\xi), \qquad S = S(\xi). \tag{5.30}$$

Equation (5.28) is a condition of compatibility of Eqs. (5.29) for the function τ. Relations (5.29) allow V and N to be expressed in terms of τ:

$$V = \tau_t, \qquad N = H_0M^{-1}(\ln|\tau|)_\xi. \tag{5.31}$$

Substitution of (5.30) and (5.31) into (5.26) leads to the following equation for τ:

$$f(\xi)\tau_{tt} = H_0^2(M^{-1}(\ln|\tau|)_\xi)_\xi. \tag{5.32}$$

Using the assumption of a linear dependence $U(x)$, which in Lagrangian coordinates is equivalent to relations (5.7), we search the function τ in the form

$$\tau = \alpha(t)\beta(\xi).$$

Relation (5.32) leads to

$$M(t)f(\xi)\ddot\alpha(t)\beta(\xi) = H_0^2(\ln|\beta(\xi)|)'' \tag{5.33}$$

(as above, dot above a symbol denotes differentiation of the corresponding function with respect to t, and prime denotes differentiation with respect to ξ). Separating the variables in (5.33), we obtain

$$M(t)\ddot\alpha(t) = C_1, \qquad H_0^2(\ln|\beta(\xi)|)'' = C_1f(\xi)\beta(\xi). \tag{5.34}$$

By virtue of the above assumptions, Eq. (5.25) becomes

$$\ddot M(t)\xi + \frac{\alpha(t)\beta(\xi)}{f(\xi)}\Big(\frac{1}{M^\gamma(t)\alpha^\gamma(t)}\Big(S(\xi)\,\frac{f^\gamma(\xi)}{\beta^\gamma(\xi)}\Big)' + \frac{H_0^2(\ln|\beta(\xi)|)'(\ln|\beta(\xi)|)''}{M^2(t)}\Big) = 0. \tag{5.35}$$

In Eq. (5.35), the variables are separated according to the Ovsyannikov lemma. The following case is nontrivial ($M \neq \text{const}$):

$$\frac{\beta}{f}\Big(S\,\frac{f^\gamma}{\beta^\gamma}\Big)' = C_3\xi, \qquad \frac{\beta(\ln|\beta|)'(\ln|\beta|)''}{f} = C_3\xi; \tag{5.36}$$

$$\ddot M + \frac{C_2}{M^\gamma\alpha^{\gamma-1}} + \frac{H_0^2C_3\alpha}{M^2} = 0. \tag{5.37}$$

The second equations of (5.34) and (5.36) lead to

$$\frac{C_1f(\xi)\beta(\xi)}{H_0^2} = \frac{C_3\xi f(\xi)}{\beta'(\xi)} \qquad \Longrightarrow \qquad \beta^2 = C_1^{-1}C_3H_0^2\xi^2 + C_4. \tag{5.38}$$

As in the previous model, in order for the solution to be physically meaningful, it is necessary that the inequality $C_1C_3 > 0$ be satisfied. The function f is given by

$$f(\xi) = \frac{H_0^2(\ln|\beta(\xi)|)''}{C_1\beta(\xi)}.$$

Integration of the first equation of (5.36), in view of (5.38), yields

$$S(\xi)\,\frac{f^\gamma(\xi)}{\beta^\gamma(\xi)} = \frac{C_2C_3H_0^4\xi^2}{2(C_1C_4 + C_3H_0^2\xi^2)^2} + C_5.$$

Then,

$$p = M^{-\gamma}\alpha^{-\gamma}\Big(p_0 + \frac{C_2C_3H_0^4\xi^2}{2(C_1C_4 + C_3H_0^2\xi^2)^2}\Big),$$

and the density is defined as

$$\rho = \frac{f(\xi)}{M(t)\alpha(t)\beta(\xi)} = M^{-1}\alpha^{-1}\,\frac{C_3H_0^4(C_1C_4 - C_3H_0^2\xi^2)}{(C_3H_0^2\xi^2 + C_1C_4)^3}. \tag{5.39}$$

Finally, using Eqs. (5.34) and (5.37), for the functions M and α, we obtain

$$\ddot{M} = -\frac{C_2}{M^\gamma\alpha^{\gamma-1}} - \frac{H_0^2C_3\alpha}{M^2}, \qquad \ddot{\alpha} = \frac{C_1}{M}. \tag{5.40}$$

For $\xi^2 = C_1C_4C_3^{-1}H_0^{-2}$, expression (5.39) for the density vanishes. In order that the expression for the pressure also vanish for the given value of ξ, it is necessary that the following the condition be satisfied:

$$p_0 = -\frac{C_2H_0^2}{8C_1C_4}.$$

In this case, the expression for the pressure becomes

$$p = -\frac{C_2H_0^2(C_3H_0^2\xi^2 - C_1C_4)^2}{8M^\gamma\alpha^\gamma C_1C_4(C_3H_0^2\xi^2 + C_1C_4)^2}.$$

The signs of the constants included in the solution are chosen so that, for $\xi = 0$, the functions β and ρ, p are defined and positive:

$$C_4 > 0, \qquad C_3 > 0, \qquad C_1C_2 < 0.$$

Using the previously obtained inequality $C_1C_3 > 0$, we have

$$C_1 > 0, \qquad C_2 < 0, \qquad C_3 > 0, \qquad C_4 > 0. \tag{5.41}$$

If inequalities (5.41) are satisfied, the obtained solution describes the evolution of a plane ideal-plasma layer adjacent to vacuum. We note that, in this solution, the particle trajectories and magnetic lines are also determined from formulas (5.18)–(5.20). This study results in the following theorem.

Theorem 3. *For system* (1.3)*, the solution with a linear dependence $U(t)$ is given by the formulas*

$$U = \frac{\dot{M}(t)}{M(t)}\,x, \qquad V = \dot{\alpha}(t)\beta(\xi), \qquad N = \frac{C_3H_0^3\xi}{M(t)(C_3H_0^2\xi^2 + C_1C_4)},$$

$$p = -\frac{C_2H_0^2(C_3H_0^2\xi^2 - C_1C_4)^2}{8M(t)^\gamma\alpha(t)^\gamma C_1C_4(C_3H_0^2\xi^2 + C_1C_4)^2}, \qquad \rho = \frac{C_3H_0^4(C_1C_4 - C_3H_0^2\xi^2)}{M(t)\alpha(t)(C_3H_0^2\xi^2 + C_1C_4)^3},$$

$$\tau = \alpha(t)\beta(\xi), \qquad \beta = \sqrt{C_1^{-1}C_3H_0^2\xi^2 + C_4}, \qquad \xi = xM(t)^{-1}$$

with arbitrary constants C_i satisfying inequalities (5.41)*. The functions M and α are found by solving the ordinary differential equations* (5.40)*. The magnetic lines are hyperbolas are determined from formulas* (2.4) *and* (5.20)*. The particle trajectories are given by Eqs.* (2.3)*, where the dependence $l(t)$ is defined in* (5.19)*.*

6. Case of an Ideal Fluid ($H \equiv 0$ and $\rho = 1$). We examine only the case $h \neq 0$ because for $h = 0$ the solution is trivial. The invariant system (1.3) is simplified to

$$\tau U_x + V = 0, \qquad U_t + UU_x + \rho^{-1} p_x = 0,$$
$$V_t + UV_x = 0, \qquad \tau_t + (\tau U)_x = 0. \tag{6.1}$$

The noninvariant function ω is given by the implicit equation

$$F(\xi, y - \tau\cos\omega, z - \tau\sin\omega) = 0,$$

where ξ is an arbitrary function which satisfies the equation $\xi_t + U\xi_x = 0$. Below, it is shown that Eqs. (6.1) can be completely integrated in the Lagrangian coordinates (t, ξ). For convenience, we use the function $\tau_1 = \tau - tV$, which, by virtue of (6.1), satisfies the equation $\tilde{D}\tau_1 = 0$. Thus, in the Lagrangian coordinates, system (6.1) has two integrals, which are written as

$$V = V(\xi), \qquad \tau_1 = G(\xi)V(\xi)$$

(V and G are arbitrary functions). Using the function $M = \partial x/\partial\xi$, we write the first equation (6.1) as

$$V(\xi)(t + G(\xi))M^{-1}M_t + V(\xi) = 0.$$

Under the assumption that $V(\xi) \neq 0$, this equation is integrated with an arbitrary function F:

$$M = F(\xi)/(t + G(\xi)). \tag{6.2}$$

For the dependences $x = x(t, \xi)$ and $p = p(t, \xi)$, we obtain the system

$$x_\xi = F(\xi)(t + G(\xi))^{-1}; \tag{6.3}$$

$$p_\xi = x_{tt}F(\xi)(t + G(\xi))^{-1}. \tag{6.4}$$

Since the Lagrangian coordinate is chosen arbitrarily, it can be assumed that $F(\xi) \equiv 1$. Integration of Eq. (6.3) with respect to ξ with the given function G yields the dependence $x(t, x_0)$ along the particle trajectory. Substitution of this dependence into Eq. (6.4) and integration of the result gives the pressure p along the particle trajectories. We note that the additive functions of time which arise from the integration of Eqs. (6.4) and (6.3) can be considered zero by virtue of the infinite-dimensional group of transformations admitted by Eqs. (6.1), which is usual for the ideal fluid equations.

The stationary solution of system (6.1) is given by the explicit formulas

$$U = U_0\,\mathrm{e}^{-mV_0x}, \qquad V = V_0, \qquad \tau = (mU_0)^{-1}\,\mathrm{e}^{mV_0x}, \qquad p = p_0 - (1/2)\rho\,U_0^2\,\mathrm{e}^{-2mV_0x},$$

where U_0, V_0, m, p_0, and ρ are arbitrary constants. The streamline pattern is given by the dependence $l(x)$ in the form

$$l(x) = (mU_0)^{-1}(\mathrm{e}^{mV_0x} - \mathrm{e}^{mV_0x_0}). \tag{6.5}$$

The exponential curves (6.5) attached to each point in the plane $x = x_0$ according to the direction fields given by the implicit equations (1.4) form the fluid flow pattern over the entire domain of the solution.

Conclusions. The properties of the submodel [1] of the equations of ideal magnetohydrodynamics that describes a generalization of the classical one-dimensional plasma flow with plane waves were studied. In the plasma flow defined by the submodel, the particle trajectories and magnetic lines were shown to be plane curves. The trajectory of each particle and the magnetic line through this particle at each fixed time lie entirely in the same plane parallel to the Ox axis. Unlike in the classical one-dimensional solution, in this solution, the plane of motion of each particle has its own orientation given by a certain additional finite relation. The functional arbitrariness available in the relation allows the geometry of the motion to be changed according to the problem solved. Exact solutions of the submodel are found that specify motion with uniform deformation along the Ox axis.

This work was supported by the Russian Foundation of Basic Research (Grant No. 05-01-00080), Integration Project of the Siberian Division of the Russian Academy of Sciences No. 2.15, and Council on Grants of the President of the Russian Federation on the Support of Young Russian Scientists and Leading Scientific Schools (Grant Nos. NSh-5245.2006.1 and MK-1521.2007.1).

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